Time Series Exercise Sheet 11

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Exercise 11.1

Consider two LTI filters L_1 and L_2, and let $\alpha \in \mathbb{C}$ then the filter

$$L = \alpha L_1 + L_2 \tag{1}$$

is also an LTI filter.

Exercise 11.2

Consider two LTI filters L_1 and L_2. The filter $L = L_1 L_2$, i.e. so that

$$L\left[\{X_t\}\right] = L_1\left[L_2\left[\{X_t\}\right]\right] \tag{2}$$

is also an LTI filter.

Exercise 11.3

A digital filter L is an LTI filter if and only if we can write the filter output as a convolution:

$$L\left[\{X_t\}\right]_u = \Delta \sum_{m \in \mathcal{T}} h_{u-m} X_m \tag{3}$$

for any $u \in \mathcal{T}$.

Exercise 11.4

Consider a stationary mean-zero time series $\{X_t\}$, with spectral representation

$$X_t = \int_{-1/2\Delta}^{1/2\Delta} e^{2\pi i f t} \mathrm{d}Z(f). \tag{4}$$

Assume that we observe this time series at the points $T = \{\Delta, \ldots, \Delta n\}$, and we define the tapered discrete Fourier transform by

$$J_h(f) = \sum_{t \in T} h_t X_t e^{-2\pi i f t} \tag{5}$$

where $\|h_t\|_2^2 = 1$ (we assume the mean is known to be zero, so do no mean correction). Show that

$$J_h(f) = \frac{1}{\Delta} \int_{-1/2\Delta}^{1/2\Delta} H(f - f') \mathrm{d}Z(f'). \tag{6}$$

Exercise 11.5

If $\{X_t\}$ is a stationary series, define $\{Y_t\} = (\mathrm{I} - \mathrm{B})[\{X_t\}]$. Is Y_t stationary? If so, what is the spectral density function of $\{Y_t\}$ in terms of the spectral density function of $\{X_t\}$?

Exercise 11.6

For this question we fix $\Delta = 1$. In lecture 3, we claimed that AR(p) processes were time reversible. In other words, If $\{X_t\}$ is an AR(p) process, then if $\{Y_t\}$ is such that for all $t \in \mathbb{Z}$, $Y_t = X_{-t}$, then Y_t is an AR(p) process with the same parameters as X_t. Specifically, if X_t had an AR representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \epsilon_t \tag{7}$$

then

$$Y_t = \sum_{j=1}^{p} \phi_j Y_{t-j} + \tilde{\nu}_t \tag{8}$$

where $\tilde{\nu}_t$ has the same distribution as ϵ_t. Prove this result.